§ <u>The broader picture</u> (cf. Notes on Langlands by J. Anschütz)

$GL_2/\mathbb{Q}$ (easiest non-trivial red. grp. $/\mathbb{Q}$. Definitely a natural starting datum.)

In Langlands program, one is interested in passing from representation theory of $GL_2(\mathbb{A}) \circlearrowright L^2(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}))$ to that of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, and vice versa.

In ptic, one wants to relate an analytic object ($GL_2(\mathbb{R}) \subset GL_2(\mathbb{A})$ is a contained real Lie group) to algebraic objects (Galois representations).

In some sense, L^2-space should decompose into irreducibles.

<u>Obstruction</u> $GL_2(\mathbb{A})$-reps form only continuous families because one has $\det : GL_2(\mathbb{A}) \longrightarrow \mathbb{A}^\times$ (center of $GL_2(\mathbb{A})$) & characters of $\mathbb{R}_{>0}$ form continuous family

$$\{ s \longmapsto \exp(2\pi i \cdot \lambda s) \}_{s \in \mathbb{R}}$$

So consider $V := L^2(GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / \mathbb{R}_{>0})$

Next idea: Consider action $GL_2(\mathbb{A}) \supset GL_2(\mathbb{R}) \supset SO(2) \curvearrowright V$

It is a compact Lie group acting unitarily

$$\Longrightarrow \quad V = \bigoplus_{n \in \mathbb{Z}} V_n \qquad \text{weight decomposition,}$$

V_n eigenspace for character $\chi_n : SO(2) \to SO(2)$, $z \mapsto z^n$

Then each V_n may be understood as a space of $SO(2)$-invariant functions after multiplying by a function that transforms with χ_{-n}.

Furthermore: Restrict attention to $GL_2(\mathbb{A})$-reps π s.th.

$$\pi^K \neq 0 \quad \text{for some open compact } K \subseteq GL_2(\mathbb{A}_f).$$

Upshot: One is lead to consider the quotient

$$Y_K := GL_2(\mathbb{Q}) \backslash \left(GL_2(\mathbb{A}_f)/K \times GL_2(\mathbb{R})/\mathbb{R}_{>0}SO(2) \right)$$

At ∞:

$$GL_2(\mathbb{R})/\mathbb{R}_{>0}SO(2) \xrightarrow{\cong} \mathcal{H}^{\pm}$$

$$g \cdot SO(2) \longmapsto g \cdot i = \frac{ai+b}{ci+d}$$

Check: $i = \dfrac{ai+b}{ci+d} \iff -c + id = ai + b$

$$\iff \quad g = \lambda \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\text{i.e.} \in \mathbb{R}_{>0} \cdot SO(2)$$

<u>Understanding K</u>

<u>Exercise</u> $GL_2(\mathbb{Z}_p) \subseteq GL_2(\mathbb{Q}_p)$ is open + maximal compact.

(i.e. $GL_2(\mathbb{Z}_p) \subseteq K_p$ compact $\Rightarrow$ $K_p = GL_2(\mathbb{Z}_p)$.)

<u>Lemma</u> Given $K_p \subset GL_2(\mathbb{Q}_p)$ any compact,

$\exists\ g \in GL_2(\mathbb{Q}_p)$ s.th. $g K_p g^{-1} \subseteq GL_2(\mathbb{Z}_p)$

<u>Proof</u> $GL_2(\mathbb{Z}_p) = \mathrm{Stab}(\mathbb{Z}_p^2 \subseteq \mathbb{Q}_p^2)$

Any lattice $\Lambda \subset \mathbb{Q}_p^2$ of form $g \cdot \mathbb{Z}_p^2$. Then

$\mathrm{Stab}(\Lambda) = g\, GL_2(\mathbb{Z}_p)\, g^{-1}$.

Given compact K_p, pick any lattice Λ.

Since $\mathrm{Stab}(\Lambda)$ open, $K_p / \mathrm{Stab}(\Lambda)$ finite.

$$\Rightarrow \sum_{k \in K_p/\mathrm{Stab}(\Lambda)} k \cdot \Lambda =: \tilde{\Lambda} \text{ is } K_p \text{ stable}$$

$\Rightarrow$ $\exists\ g \in GL_2(\mathbb{A}_f)$ s.th. $g K g^{-1} \subseteq GL_2(\hat{\mathbb{Z}})$.

$\square$

Given $K \subseteq GL_2(\mathbb{A}_f)$, has form $K_S \times GL_2(\hat{\mathbb{Z}}^S)$

where
- S finite set of primes
- $K_S \subseteq \prod_{p \in S} GL_2(\mathbb{Q}_p)$ any open + compact subgroup
- $\hat{\mathbb{Z}}^S = \prod_{p \notin S} \mathbb{Z}_p$.

Then $K_S \subseteq \prod_{p \in S} \underbrace{\mathrm{proj}_p(K_S)}_{\text{again open compact}}$

Prev. Lemma now applies prime-by-prime

$\Rightarrow \exists\, g \in GL_2(\mathbb{A}_f)$ s.th.

$g K g^{-1} \subseteq GL_2(\hat{\mathbb{Z}})$.

Class number of GL_2 is 1.

<u>Prop</u> 1) $GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) / GL_2(\hat{\mathbb{Z}}) = \{pt\}$

2) For any K, $GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) / GL_2(\hat{\mathbb{Z}})$ is finite.

<u>Proof</u> 1) $g = (g_p)_p \in GL_2(\mathbb{A}_f)$. There is finite S s.th.

$p \notin S \Rightarrow g_p \in GL_2(\mathbb{Z}_p)$.

$$\Longrightarrow \quad g \sim (g_p)_{p \in S} \times (1)_{p \notin S}.$$

Fix $p \in S$. Elementary divisor then allows to write

$$g_p = A \begin{pmatrix} p^n & \\ & p^m \end{pmatrix} B \quad , \quad A, B \in GL_2(\mathbb{Z}_p)$$

$$n, m \in \mathbb{Z}.$$

Replacement possible:

$$A \longmapsto A \begin{pmatrix} \det(A)^{-1} & \\ & 1 \end{pmatrix}$$

$$B \longmapsto \begin{pmatrix} \det(A) & \\ & 1 \end{pmatrix} B$$

So wlog, $\det A = 1$.

Then $\forall r \geq 1$, $\exists h \in GL_2(\mathbb{Z})$ s.th. $h \equiv A \mod p^r$.

(Seen before, euclidean algorithm.)

Furthermore, conjugation action of $\begin{pmatrix} p^n & \\ & p^m \end{pmatrix}$ on $GL_2(\mathbb{Q}_p)$ continuous

$$\Longrightarrow \exists r \geq 1 \text{ s.th. } \underbrace{T \equiv 1 \mod p^r}_{\text{Conditions define nbhd basis of } 1 \in GL_2(\mathbb{Z}_p)}$$

$$\longrightarrow \begin{pmatrix} p^n & \\ & p^m \end{pmatrix}^{-1} T \begin{pmatrix} p^n & \\ & p^m \end{pmatrix} \in GL_2(\mathbb{Z}_p).$$

Pick h for this r.

Then

$$g_p \sim \underbrace{\begin{pmatrix} p^{-n} & \\ & p^{-m} \end{pmatrix} h^{-1}}_{\in GL_2(\mathbb{Q})} \cdot g_p$$

$$= \begin{pmatrix} p^{-n} & \\ & p^{-m} \end{pmatrix} h^{-1} A \begin{pmatrix} p^{n} & \\ & p^{m} \end{pmatrix} B \quad \in GL_2(\mathbb{Z}_p)$$

Moreover, $\begin{pmatrix} p^{-n} & \\ & p^{-m} \end{pmatrix} h^{-1} \in GL_2(\mathbb{Z}_\ell) \quad \forall \ell \neq p$

Now induct on $\# S$. $\square$ 1)

2) $GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) / K \xrightarrow{\cong} GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) / gKg^{-1}$ $(\ast)$

$[h] \longmapsto [hg^{-1}]$

So wlog, $K \subseteq GL_2(\hat{\mathbb{Z}})$.

Since open & $GL_2(\hat{\mathbb{Z}})$ compact, $GL_2(\hat{\mathbb{Z}})/K$ finite.

$\Longrightarrow \square$ 2)

We obtain following description of Y_K:

$g_1, \dots, g_r \in GL_2(\mathbb{A}_f)$ representatives for $GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) / K$.

$$Y_K = \coprod_{\Gamma_i} \Gamma_i \backslash \mathcal{H}^{\pm}, \quad \Gamma_i = GL_2(\mathbb{Q}) \cap g_i K g_i^{-1}$$

If $K \subseteq GL_2(\hat{\mathbb{Z}})$, then may choose $g_i \in GL_2(\hat{\mathbb{Z}})$.

Moreover, $\exists\, N \geq 1$ s.th. $K(N) := \ker\left(GL_2(\hat{\mathbb{Z}}) \longrightarrow GL_2(\mathbb{Z}/N)\right) \subseteq K$.

$$\Longrightarrow \{\gamma \mid \gamma \equiv 1 \bmod N\} \subseteq \Gamma_i \subseteq GL_2(\mathbb{Z}).$$

so Γ's are congruence subgroups.

<u>In fact</u> This IS the general case:

There is an iso
$$Y_K \xrightarrow[\cdot g^{-1}]{\cong} Y_{gKg^{-1}}$$
$$[h, \tau] \longmapsto [hg^{-1}, \tau]$$

so may always reduce to $K \subseteq GL_2(\hat{\mathbb{Z}})$.

<u>Example</u> $GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) / K(N)$

$$= GL_2(\mathbb{Z}) \backslash GL_2(\hat{\mathbb{Z}}) / K(N) = GL_2(\mathbb{Z}) \backslash GL_2(\mathbb{Z}/N)$$

$$\xrightarrow[\det]{\sim} (\mathbb{Z}/N)^\times/\{\pm 1\}$$

$K(N) \subseteq GL_2(\hat{\mathbb{Z}})$ normal $\Rightarrow$ for all i,

$$\Gamma_i = \Gamma(N) := \{\gamma \mid \gamma \equiv 1 \mod N\} \subseteq GL_2(\mathbb{Z})$$

$$Y_{K(N)} = \coprod_{(\mathbb{Z}/N)^\times/\{\pm 1\}} \Gamma(N) \backslash \mathcal{H}^\pm$$

Now assume $K(N) \subseteq K \subseteq GL_2(\hat{\mathbb{Z}})$.

$K(N)$ is normal in $GL_2(\hat{\mathbb{Z}})$, so also in K.

$\Rightarrow$ Obtain finite group K/K_N that acts:

$$Y_K = Y_{K(N)} / (K/K(N)).$$

$\mathcal{H}^\pm$ has complex structure & $GL_2(\mathbb{R}) \curvearrowright \mathcal{H}^\pm$ holomorphically

$\Rightarrow$ Each Y_K Riemann surface.

Analytic theory of j-invariant (cf. last term §3)

$\Rightarrow$ Even $\mathbb{C}$-points of affine algebraic curve$/\mathbb{C}$.

Thm There is an inverse system of affine curves / Spec $\mathbb{Q}$

$\{M_K\}_{K \subset GL_2(\mathbb{A}_f)}$ together with a $GL_2(\mathbb{A}_f)$-action

$\{ g^{-1} : M_K \xrightarrow{\cong} M_{gKg^{-1}} \}$ s.th. its $\mathbb{C}$-points

are isomorphic to $\{ Y_K , g^{-1} : Y_K \xrightarrow{\cong} Y_{gKg^{-1}} \}$.

Sketch M_{K_N} := moduli of ECs + level-N-str.

$(N \geq 3)$

For $K_N \subseteq K \subseteq GL_2(\hat{\mathbb{Z}})$, $M_K := M_{K_N} / (K / K_N)$

(Here $GL_2(\hat{\mathbb{Z}}) \curvearrowright M_{K_N}$ via $GL_2(\hat{\mathbb{Z}}) \to GL_2(\mathbb{Z}/N)$.)

For general K, pass to quasi-isogeny moduli problem

(The functorialities between the M_K will be discussed

in more detail later, so we'll stop here.) $\square$

Terminology $\{M_K\}$ is the Shimura variety

for GL_2.

Moduli space provides the

canonical model for the $\{Y_K\}$.

<u>Upshot</u> Have achieved a passage from analytic setting to a number-theoretic one.

To go further, need to also understand integral models at primes dividing the level.